Exc 1 $f(x) = \sum_{i=0}^{\infty} a_i \phi_i(x)$ ϕ_i orthogonal polynomials in some inner product

a) show $a_i = (f,\phi_i)/(\phi_i,\phi_i)$

$$(f,\phi_j) = \left(\sum_{i=0}^{\infty} a_i\phi_i , \phi_j\right) = \sum_{i=0}^{\infty} a_i (\phi_i,\phi_j)$$

$$= a_j (\phi_j,\phi_j)$$

since $(\phi_i,\phi_j)=0$ $i\neq j$

$$\Rightarrow a_j = (f,\phi_j)/(\phi_j,\phi_j)$$

b) show $\sum_{i=0}^{\infty} a_i^2 (\phi_i,\phi_i) = (f,f)$, what is name and meaning of this expression?

- $(f,f) = \left(\sum_{i=0}^{\infty} a_i\phi_i(x), \sum_{j=0}^{\infty} a_j\phi_j(x)\right)$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i a_j (\phi_i,\phi_j)$$

$$= \sum_{i=0}^{\infty} a_i^2 (\phi_i,\phi_i)$$ since $(\phi_i,\phi_j)=0$ $i\neq j$

- this is Parseval relation
- means 2-norm of f is equal to a weighted infinite vector norm over its coefficients

c) assume $f(x) = p_n(x)$ p_n polynomial of degree n
show $a_i = 0$ for $i > n$

- $p_n(x)$ can be written as $\sum_{i=0}^{n} c_i\phi_i(x)$
- given $f(x) = \sum_{i=0}^{\infty} a_i\phi_i(x) = p_n(x) = \sum_{i=0}^{n} c_i\phi_i(x)$

$$\Rightarrow \sum_{i=0}^{n}(a_i-c_i)\phi_i(x) + \sum_{i=n+1}^{\infty} a_i\phi_i(x) = 0$$

$$\Rightarrow \sum_{i=0}^{n}(a_i-c_i)(\phi_i,\phi_j) + \sum_{i=n+1}^{\infty} a_i(\phi_i,\phi_j) = 0 \quad j=0,1,\dots$$

$$\Rightarrow a_j(\phi_j,\phi_j) = 0 \quad \text{if } j=n+1,\dots$$

$$\Rightarrow a_j = 0 \quad j>n$$

d) what will error be if we approximate $f(x) = p_n(x)$ by the first $n-1$ terms of the expansion in orthogonal polynomials

- from c) $f(x) = \sum_{i=1}^{\infty} a_i \phi_i(x)$ expansion in orthogonal pol.

$a_i = 0 \quad i > n \quad$ if $f(x) = p_n(x)$

$$\Rightarrow f(x) = \sum_{i=1}^{n} a_i \phi_i(x) = \sum_{i=1}^{n-1} a_i \phi_i(x) + a_n \phi_n(x)$$

$\Rightarrow$ if we approximate by first $n-1$ terms of the expansion than error $=$ $a_n \phi_n(x)$

e) suppose $f(x)$ defined on finite interval for which orthogonal polynomials ϕ_i will the approximation in the previous part give us the minimax approximation i.e the best polynomial approximation. Explain why.

- minimax approximation is found if the error satisfies the equioscillation theorem, i.e all the $n+1$ extrema of the error should be equally big and alternating in sign.

- from d) : error $= a_n \phi_n(x)$ $\quad \phi_n$ polynomial

hence ϕ_n should be such that it oscillates, and $n+1$ extrema are equally big and alternate in sign

$\Rightarrow \phi_n$ Chebyshev polynomial

- since Chebyshev polynomials are defined on $[-1, 1]$

$\Rightarrow \phi_i$ Chebyshev polynomials shifted to the interval on which $f(x)$ is defined

f) suppose a_i decrease rapidly with i. Based on the previous parts, show that the same orthogonal polynomials as in part e) are the best choice to find least squares approximations to the minimax approximation

- if a_i decreases rapidly and f is smooth then using Chebyshev pol. in least squares will give us an error which almost satisfies the equioscillation theorem
- hence we are close to minimax solution

Exc 2. $\int_0^\infty \exp(-x)\, f(x)\, dx$

a. using orthogonal polynomials derive that for this integral the Gauss rule using one interpolation point is simply $f(1)$

- inner product $(f, g) = \int_0^\infty \exp(-x) \cdot f(x)\, g(x)\, dx$
- one interpolation point $\Rightarrow$ we need first order-orthogonal polynomial
- $\varphi_0(x) = 1$

  $\varphi_1(x) = x - \alpha$ with α s.t $(\varphi_1, \varphi_0) = 0$

  $(\varphi_1, \varphi_0) = \int_0^\infty \exp(-x)(x - \alpha)\, dx = 0$

  $$\Rightarrow \alpha = \frac{\int_0^\infty x \exp(-x)\, dx}{\int_0^\infty \exp(-x)\, dx}$$

  $\int_0^\infty \exp(-x)\, dx = -\exp(-x)\Big|_0^\infty = 1$

  $\int_0^\infty \underbrace{x}_{f}\, \underbrace{\exp(-x)}_{g'}\, dx \underset{p.i.}{=} -x\exp(-x)\Big|_0^\infty - \int_0^\infty 1 \cdot -\exp(-x)\, dx$

  $= + \int_0^\infty \exp(-x)\, dx = 1$

  $\Rightarrow \alpha = \frac{1}{1} = 1 \qquad \Rightarrow \varphi_1(x) = x - 1$

- Gauss rule : $w_0 f(x_0)$ with x_0 zero of $\varphi_1(x)$

  $\Rightarrow x_0 = 1$

  $w_0 = \int_0^\infty l_0(x) \exp(-x)\, dx$

  $l_0(x) = 1 \Rightarrow w_0 = \int_0^\infty \exp(-x)\, dx = 1$

  $\Rightarrow$ Gauss rule : $f(1)$

b. - if polynomial orthogonal is of degree n than Gauss rule degree of exactness $2n - 1$

- now $n = 1 \Rightarrow$ degree of exactness $2 \cdot 1 - 1 = 1$
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	Gauss rule $f(1)$	$\int_0^\infty \exp(-x) f(x)\, dx$	$\Rightarrow$ agree for $1, x$ not agree for x^2 $\Rightarrow$ degree exactness indeed 1
1	1	$\int_0^\infty \exp(-x)\, dx = 1$	✓
x	1	$\int_0^\infty \exp(-x)\, x\, dx = 1$	✓
x^2	1	$\int_0^\infty \exp(-x) x^2\, dx = -x^2\exp(-x)\Big|_0^\infty + 2\int_0^\infty x\exp(-x)\, dx = 2$	✗